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Inversive Planes of Odd Order

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Given an inversive plane we define a graph, the vertices of which are the blocks of the inversive plane, where two blocks are adjacent if they meet in one point. We use this graph to provide the following characterization of the classical inversive planes of odd order. An inversive plane of odd order is classical if, and only if, there is a point at which the residue is an affine plane over a field, and the graph defined above has two connected components.

1. Introduction

An *inversive plane of order* q is a $3-(q^2+1, q+1, 1)$ design. Following Dembowski [5] we shall call the blocks of an inversive plane *circles*. There are two known infinite series of finite inversive planes, both of which are constructed from ovoids of $PG(3, q)$ by taking the points to be the points of the ovoid and the circles to be the secant plane sections; such inversive planes are called *egglike* ([5], §6.1.3). They are denoted $\mathcal{M}(q)$ (constructed from the elliptic quadrics, one for each prime power q) and $\mathcal{S}(q)$ (constructed from the Tits ovoids (see [11]), one for each $q = 2^e$, where $e > 1$ is odd). We shall call the inversive planes $\mathcal{M}(q)$ *classical* inversive planes. It was proved by Dembowski [4] that all inversive planes of even order are egglike, and we shall be considering the problem of whether all inversive planes of odd order must also be egglike. It has been proved that $\mathcal{M}(3)$, $\mathcal{M}(5)$ and $\mathcal{M}(7)$ are the only inversive planes of order 3, 5 and 7 respectively ([3, 6, 7, 13]).

Given an inversive plane $\mathcal{I}$ of order q we define a graph $\Gamma(\mathcal{I})$ as follows: the vertices of $\Gamma(\mathcal{I})$ are the circles of $\mathcal{I}$ and two circles are adjacent if they intersect in precisely 1 point (notice that two circles intersect in 0, 1 or 2 points). In Theorem 3.1 we demonstrate that $\Gamma(\mathcal{I})$ has 1 connected component if q is even, and 1 or 2 connected components if q is odd. Specializing to the case q odd, it is shown that the graphs of the known inversive planes have 2 components. The main result of this paper provides a partial converse to this result and hence a characterization of the classical inversive planes of odd order. The following theorem is proved as Theorem 3.3 (terms such as residue are defined in Section 2).

Main Theorem. *An inversive plane $\mathcal{I}$ of odd order q is isomorphic to $\mathcal{M}(q)$ iff $\Gamma(\mathcal{I})$ has 2 connected components and there is some point P such that the residue of $\mathcal{I}$ at P is isomorphic to $AG(2, q)$.*

As a corollary to this theorem it is shown that for odd order classical inversive planes each of the components is a 2-design. Furthermore, we note that for odd $q > 3$ each of the components of $\Gamma(\mathcal{M}(q))$ forms a 3-class association scheme, whereas for even $q > 2$ all the circles of an inversive plane form such a scheme.

2. Preliminary Results

Throughout this paper we use the terminology and notation of Dembowski [5], with the exception of the following terms. A $t-(v, k, \lambda)$ design is an incidence structure on v points, where each block contains k points and any t-subset of the points lies on λ blocks. Given an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ and a point $P \in \mathcal{P}$ the *residue of* I *at*

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P is the incidence structure $\mathcal{I}_P = (\mathcal{P}', \mathcal{L}', I')$, where $\mathcal{P}' = \mathcal{P} - \{P\}$, $\mathcal{L}' = \{L \in \mathcal{L} \mid (P, L) \in I\}$ and $I' = I \cap (\mathcal{P}' \times \mathcal{L}')$. Less formally, the residue $\mathcal{I}_P$ is the incidence structure containing only the blocks through the point P, but with P deleted.

It is straightforward to show that the residue $\mathcal{I}_P$ of the inversive plane $\mathcal{I}$ at the point P is a $2-(q^2, q, 1)$ design—an affine plane of order q. The residue at every point of $\mathcal{M}(q)$ and $\mathcal{S}(q)$ is the affine plane over a field $AG(2, q)$. We shall say that the inversive plane $\mathcal{I}$ is a *one-point extension* of the affine plane $\mathcal{A}$ if there is some point P such that $\mathcal{I}_P = \mathcal{A}$.

An *oval* of an affine plane $\mathcal{A}$ of order q is a set of $q + 1$ points with no three collinear.

LEMMA 2.1. *Let l be a circle of the inversive plane $\mathcal{I}$ and P a point not incident with l. Then the set of points on l is an oval of the affine plane $\mathcal{I}_P$.*

PROOF (see also [12]). As any three points of $\mathcal{I}$ lie on a unique circle any two circles can intersect in at most two points. Suppose for a contradiction that three of the points of l are collinear in $\mathcal{I}_P$. Then by the definition of residue, these three points lie on a circle of $\mathcal{I}$ containing P. Thus these three points are on two circles of $\mathcal{I}$, which is the required contradiction. □

Therefore the circles of $\mathcal{I}$ fall into two groups: those incident with P which form the lines of the affine plane $\mathcal{I}_P$, and those not incident with P which form ovals in the affine plane. We shall often abuse our terminology slightly by referring to circles of $\mathcal{I}$ incident with P and lines of $\mathcal{I}_P$ interchangeably, although they are actually in different incidence structures. The context in which this is done will make the meaning clear.

A *conic* of $PG(2, q)$ is the set of singular one-dimensional subspaces of a non-degenerate orthogonal space $(GF(q)^3, Q)$. Embedding $AG(2, q)$ in the usual way in $PG(2, q)$ with line at infinity l, an *ellipse* of $AG(2, q)$ is a conic for which l is an external line (that is, a line that misses the conic).

The next theorem indicates why we are interested in ellipses.

THEOREM 2.2 (Segre's Theorem). *In $PG(2, q)$ with q odd, every oval is a conic and conversely. In particular, every oval of $AG(2, q)$ is an ellipse.*

PROOF. See [9], Section 8.2.4. □

THEOREM 2.3. *The group $AGL(2, q)$ is transitive on the ellipses of $AG(2, q)$. There are $q^3(q-1)^2/2$ ellipses in $AG(2, q)$.*

PROOF. We sketch a proof for odd q. Embed $AG(2, q)$ in $PG(2, q)$ in the usual way with line at infinity l. Hughes and Piper ([10], Theorem 2.60) show that $PGL(3, q)$ is transitive on conic/external line pairs. Thus the stabilizer in $PGL(3, q)$ of the line at infinity l, namely $AGL(2, q)$, is transitive on conics to which l is external; that is, the ellipses of $AG(2, q)$. By Hirschfeld ([9], Section 7.2) there are $q^5 - q^2$ conics in $PG(2, q)$, each with $q(q-1)/2$ external lines. It follows that each line is external to $(q^5 - q^2)q(q-1)/2(q^2 + q + 1) = q^3(q-1)^2/2$ conics. □

3. MAIN RESULTS

In this section we prove our main results concerning the graph $\Gamma(\mathcal{I})$ defined in the Introduction.

The following comments all follows directly from Dembowski ([5], Section 3.2.23; see also [9]). Consider the tangents to an oval $\mathcal{O}$ in an affine plane. If q is even then all the $q+1$ tangents to $\mathcal{O}$ are concurrent (meeting in a point called the *nucleus* of $\mathcal{O}$), and hence one member of each parallel class is tangent to $\mathcal{O}$. Alternatively, if q is odd then we embed the affine plane into a projective plane in the usual way and consider $\mathcal{O}$ in the projective plane. Any point not on $\mathcal{O}$ is on either 0 or 2 tangents, and in particular as all $q+1$ tangents to $\mathcal{O}$ meet the line at infinity l, exactly $(q+1)/2$ of the points of l lie on 2 tangents. Then considering the points of l to be the parallel classes of the affine plane it is clear that $(q+1)/2$ parallel classes contain no tangents to $\mathcal{O}$ and $(q+1)/2$ parallel classes contain 2 tangents to $\mathcal{O}$. We shall call a parallel class *internal* to $\mathcal{O}$ if it contains no tangents, otherwise *external*. This terminology is suggested by an interior (exterior) point to an oval being a point on zero (two) tangents.

THEOREM 3.1. *Let $\mathcal{I}$ be an inversive plane of order q. If q is even then $\Gamma(\mathcal{I})$ is connected of diameter* 2. *If q is odd then $\Gamma(\mathcal{I})$ has* 1 *or* 2 *connected components.*

PROOF. If $q=2$ then $\Gamma(\mathcal{I})$ is the Petersen graph and the result is true, so assume that $q>2$.

Suppose that q is even and consider two circles of $\mathcal{I}$. Then there is a point P not on either circle. Regarding the two circles as ovals of the affine plane $\mathcal{I}_P$, consider the line l joining the nucleii of the two ovals (or if they have a common nucleus choose l to be any line through it). Then l extended by P is a circle of $\mathcal{I}$ (by the definition of residue) and it is tangent to both circles and hence they are at distance at most 2 in $\Gamma(\mathcal{I})$.

Suppose now that q is odd and fix a point P. Then we can divide the vertices of $\Gamma(\mathcal{I})$ into ovals and lines (of $\mathcal{I}_P$). Clearly, a line is adjacent to all the lines in its own parallel class (as they meet in the one point P), and thus if two ovals have an external parallel class in common then we can find a path (in $\Gamma(\mathcal{I})$) of length at most 3 between them and hence they are in the same connected component. Each component of $\Gamma(\mathcal{I})$ contains at least one oval and its $(q+1)/2$ external parallel classes must be disjoint from those of any oval in any other connected component. Therefore $\Gamma(\mathcal{I})$ has at most 2 connected components. □

THEOREM 3.2. *Let $\mathcal{E}$ be an ellipse in $AG(2, q)$ for q odd. The number of ellipses having the same set of external parallel classes as $\mathcal{E}$ is $q^2(q-1)/2$.*

PROOF. Given an ellipse $\mathcal{E}$, let $\mathcal{S}(\mathcal{E})$ be the set of ellipses of $AG(2, q)$ with the same set of external parallel classes as $\mathcal{E}$. As $AGL(2, q)$ is transitive on ellipses it is transitive on the set $S=\{\mathcal{S}(\mathcal{E}) \mid \mathcal{E} \text{ an ellipse}\}$ and each of the members of S has the same size. We shall find the number of sets in S by finding the order of the stabilizer $G=AGL(2, q)_{\mathcal{S}(\mathcal{E})}$. Firstly we shall determine a lower bound for the order of G. collineations that fix the parallel classes (setwise) are called dilatations and form a normal subgroup D of $AGL(2, q)$ of order $q^2(q-1)$. The dilations are clearly in G, as is the stabilizer of $\mathcal{E}$ which has order $2(q+1)$. The intersection $D \cap AGL(2, q)_{\mathcal{E}}$ has order at most 2, and thus $|G| \geq q^2(q-1)(q+1)$. Now we shall determine an upper bound on the order of G by considering the completion of $AG(2, q)$ to a projective plane with line at infinity l. Then each point on l corresponds to one parallel class of $AG(2, q)$. Consider the group H that fixes the set of external parallel classes of $\mathcal{E}$ and look at its action on l. As $H \geq AGL(2, q)_{\mathcal{E}}$, $|H^l| \geq q+1$, and it has 2 orbits of length $(q+1)/2$. Now $AGL(2, q)$ induces $PGL(2, q)$ on l, and by Dickson [8] the only proper subgroup of $PGL(2, q)$ not in $PSL(2, q)$ containing H^l is dihedral of order $2(q+1)$ and transitive (hence not fixing the set of external parallel classes). Thus $|H^l| = q+1$, and $|H| = |D|\,|H'| = q^2(q-1)(q+1)$ (as D is the kernel of the homomorphism $H \to H'$). However, it is clear that $G \leqslant H$ and thus $|G| = q^2(q^2-1)$. Hence there are $[AGL(2, q): G] = q^2 - q$ sets in S and each set contains $q^3(q-1)^2/2(q^2-q) = q^2(q-1)/2$ ellipses. □

THEOREM 3.3. *Let $\mathcal{I}$ be an inversive plane of odd order q. Then $\mathcal{I} \cong \mathcal{M}(q)$ iff $\Gamma(\mathcal{I})$ has 2 connected components and there is a point P of $\mathcal{I}$ such that $\mathcal{I}_P \cong AG(2, q)$.*

PROOF. ($\Rightarrow$) We use the representation and properties of $\mathcal{M}(q)$ described in Dembowski ([5], Section 6.4); that is, the points of $\mathcal{M}(q)$ are the elements of $GF(q^2) \cup \{\infty\}$ and the circles are images of the point set $GF(q) \cup \{\infty\}$ under the group of mappings

$$v \to \frac{va + c}{vb + d}, \qquad ad \neq bc. \tag{1}$$

Consider the group Δ generated by all the mappings of the form (1) with $ad - bc$ a non-zero square in $GF(q^2)$. This group is doubly transitive on the points of $\mathcal{M}(q)$. Let $\Pi = \Delta Z$, where Z is the group of order 2 generated by the mapping $v \to v^q$. This group Π permutes the circles in 2 orbits of length $q(q^2+1)/2$, and the stabilizer of any circle is triply transitive on the points of the circle.

Considering the special circle $K = GF(q) \cup \{\infty\}$, we shall show that any circle in the same component in $\Gamma(\mathcal{I})$ as K is in the same orbit under Π, and hence that the entire component containing K is contained in one orbit of Π.

It is sufficient to show that K can be mapped to any of its neighbours C by an element σ of Π (for then C can be mapped to any of *its* neighbours by using a suitable element conjugated by σ and so on). The mappings $v \to v + c$, $c \in GF(q^2)$ are in Π (as $ad - bc = 1$), fix ∞ and are transitive on all the circles that meet K only in ∞. Thus we can map K to any of its neighbours that meet K in ∞. As Π_K is (triply) transitive on the points we can map any point to ∞ and hence by suitable conjugation can map K to any of its neighbours meeting K in any of its points. Thus $\Gamma(\mathcal{I})$ has at least two components and thus by Theorem 3.1, exactly two components. As $\Gamma(\mathcal{I})$ is a vertex-transitive graph, both components have the same size $q(q^2+1)/2$ and hence each of the orbits of Π forms a component.

($\Leftarrow$) Fix the point P and divide the circles into ovals and lines of $\mathcal{I}_P (\cong AG(2, q))$. Any two ovals in the same component have the same set of external parallel classes and thus, by Theorem 3.2, a component must consist of all the ovals with a given set of external parallel classes together with the lines of those parallel classes. Then the other component is uniquely determined as those ovals (and lines) whose external parallel classes are precisely those that are internal to the first component. By the first part of this theorem, $\mathcal{M}(q)$ has the same form, and as $AGL(2, q)$ is transitive on ovals we can represent $\mathcal{M}(q)$ as precisely the same sets of ovals and thus $\mathcal{I} \cong \mathcal{M}(q)$. □

COROLLARY. *The circles in each of the two components of $\Gamma(\mathcal{M}(q))$ form a $2-(q^2+1, q+1, (q+1)/2)$ design.*

PROOF. As Π is doubly transitive on the points, the circles in each orbit form a 2-design with some λ. Counting pairs of points yields that $\lambda = (q+1)/2$. □

REMARK. An inversive plane is block-schematic (that is, the circles carry an association scheme (see [2]) with relations being the cardinalities of intersections) iff it is of even order (P. J. Cameron, private communication).

Taking R_0 to be the identity relation, define relations R_1, R_2 and R_3 on the circles of an inversive plane as follows: $(l_1, l_2) \in R_i$ iff $|l_1 \cap l_2| = i$ for $i = 1, 2$ and $(l_1, l_2) \in R_3$ iff $l_1 \cap l_2 = \varnothing$.

Under these relations, all the circles of an inversive plane of even order $q > 2$ form a 3-class association scheme with intersection matrices (see [2])

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2-1 & 0 & 0 \\ 0 & 0 & q^2(q+1)/2 & 0 \\ 0 & 0 & 0 & q(q-1)(q-2)/2 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & q-2 & q^2/2 & q(q-2)/2 \\ 0 & q^2/2 & q^2(q+2)/4 & q^2(q-2)/4 \\ 0 & q(q-2)/2 & q^2(q-2)/4 & q(q-2)(q-4)/4 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & q-1 & (q-1)(q+2)/2 & (q-1)(q-2)/2 \\ 1 & (q-1)(q+2)/2 & q(q-1)(q+4)/4 & q(q-1)(q-2)/4 \\ 0 & (q-1)(q-2)/2 & q(q-1)(q-2)/4 & (q-1)(q-2)^2/4 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & q+1 & q(q+1)/2 & (q+1)(q-4)/2 \\ 0 & q(q+1)/2 & q^2(q+1)/4 & q(q+1)(q-2)/4 \\ 1 & (q+1)(q-4)/2 & q(q+1)(q-2)/4 & (q^3-7q^2+12q+4)/4 \end{pmatrix},$$

whereas when $q > 3$ is odd the circles in *each of the two components* of $\Gamma(\mathcal{M}(q))$ form a 3-class association scheme with intersection matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2-1 & 0 & 0 \\ 0 & 0 & q(q^2-1)/4 & 0 \\ 0 & 0 & 0 & q(q-1)(q-3)/4 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2q-2 & q(q-1)/2 & q(q-3)/2 \\ 0 & q(q-1)/2 & q(q^2-1)/8 & q(q-1)(q-3)/8 \\ 0 & q(q-3)/2 & q(q-1)(q-3)/8 & q(q-3)(q-5)/8 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2q-2 & (q^2-1)/2 & (q-1)(q-3)/2 \\ 1 & (q^2-1)/2 & (q-1)^3/8+q(q-3)/2 & (q-1)^2(q-3)/8 \\ 0 & (q-1)(q-3)/2 & (q-1)^2(q-3)/8 & (q-1)(q-3)^2/8 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2q+2 & (q^2-1)/2 & (q+1)(q-5)/2 \\ 0 & (q^2-1)/2 & (q^2-1)(q-1)/8 & (q^2-1)(q-3)/8 \\ 1 & (q+1)(q-5)/2 & (q^2-1)(q-3)/8 & (q-3)^3/8-(q-9)/2 \end{pmatrix}.$$

In each case this can be proved by direct calculation of the intersection numbers p_{ij}^k.

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REFERENCES

1. E. Artin, *Geometric Algebra,* Interscience Tracts in Pure and Applied Mathematics **3,** Interscience, New York, 1957.
2. E. Bannai and T. Ito, *Algebraic Combinatorics I, Association Schemes,* Benjamin/Cummings, Menlo Park, California, 1984.
3. Y. Chen, The Steiner system $S(3,6,26)$, *J. Geom.* **2/1** (1972), 7–28.
4. P. Dembowski, Inversive planes of even order, *Bull. Am. Math. Soc.* **69** (1963), 850–854.
5. P. Dembowski, *Finite Geometries,* Springer-Verlag, Berlin, 1968.
6. R. H. F. Denniston, Uniqueness of the inversive plane of order 5, *Manuscripta Math.* **8** (1973), 11–19.
7. R. H. F. Denniston, Uniqueness of the inversive plane of order 7, *Manuscripta Math.* **8** (1973), 21–26.
8. L. E. Dickson, *Linear Groups: with an Exposition of the Galois Field Theory,* B. G. Teubner, Leipzig, 1901 (reprinted: Dover, New York, 1958).
9. J. W. P. Hirschfeld, *Projective Geometries over Finite Fields,* Clarendon Press, Oxford, 1979.
10. D. R. Hughes and F. C. Piper, *Projective Planes,* Graduate Texts in Mathematics **6,** Springer-Verlag, New York, 1973.
11. H. Lüneberg, *Die Suzukigruppen und ihre Geometrien,* Lecture Notes in Mathematics **10,** Springer-Verlag, Berlin, 1965.
12. H. Lüneberg, *Transitive Erweiterungen endlicher Permutationsgruppen,* Lecture Notes in Mathematics **84,** Springer-Verlag, Berlin, 1969.
13. E. Witt, Über Steinersche Systeme. *Abh. Hamburg* **12** (1938), 265–275.

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Note added in proof: Since completing this work we have learned that Thas has shown that for odd $q \notin \{11, 23, 59\}$, an inversive plane $\mathcal{I}$ of order q is classical if for at least one point P, $\mathcal{I}_P = AG(2, q)$.